Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Geometry and Physics 56 (2006) 1654–1665
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

A new proof of the local regularity of the eta invariant of a Dirac operator

Raphaël Ponge

Department of Mathematics, Ohio State University, Columbus, USA

Received 26 January 2005; received in revised form 12 September 2005; accepted 15 September 2005
Available online 10 January 2006

Abstract

In this paper we use the approach of our earlier proof of the local index theorem to give a new proof of Bismut–Freed's result on the local regularity of the eta invariant of a Dirac operator in odd dimension.

MSC: 58J28; 58J35; 58J40

Keywords: Eta invariant; Heat equation methods; Pseudodifferential operators; Dirac operators

1. Introduction

The eta invariant of a selfadjoint elliptic ΨDO was introduced by Atiyah–Patodi–Singer [4] as a boundary correction to their index formula on manifolds with boundary. It is obtained as the regular value at $s = 0$ of the eta function,

$$\eta(P; s) = \operatorname{Tr} P|P|^{-(s+1)} = \int_M \eta(P; s)(x). \tag{1.1}$$

However, the residue at $s = 0$ of the local eta function $\eta(P; s)(x)$ needs not vanish (see, e.g., [14]), so it is a nontrivial fact that the regular value exists (see [5,15]). Nevertheless, global K-theoretic arguments allows us to reduce the proof to the case of a Dirac operator on an odd dimensional spin Riemannian manifold with coefficients in a Hermitian vector bundle, for which the result can be obtained by using invariant theory (see [5,16]).

Subsequently, Wodzicki [25,26] generalized the result of Atiyah–Patodi–Singer and Gilkey in the nonselfadjoint setting. More precisely, he proved that:

E-mail address: ponge@math.ohio-state.edu.

doi:10.1016/j.geomphys.2005.09.003

(i) The regular value at $s = 0$ of the zeta function $\zeta_\theta(P; s) = \operatorname{Tr} P_\theta^{-s}$ of an elliptic ΨDO is independent of the spectral cutting $\{\arg \lambda = \theta\}$ used to define P_θ^s (see [26, 1.24]);
(ii) The noncommutative residue of a ΨDO projection is always zero (see [26, 7.12]).

The original proofs of Wodzicki are quite involved, but it follows from an observation of Brüning-Lesch that Wodzicki's results can be deduced from the aforementioned result of Atiyah–Patodi–Singer and Gilkey (see [11, Lem. 2.6] and [22, Rmk. 4.5]).

Next, in the case of a Dirac operator $D\!\!\!/_{\mathcal{E}}$ on a odd spin Riemannian manifold M^n with coefficients in a Hermitian bundle $\mathcal{E}$, Bismut–Freed [10] proved in a purely analytic fashion that the local eta function $\eta(P; s)(x)$ is actually regular at $s = 0$. More precisely, by the Mellin formula for $\Re s > -1$ we have

$$D\!\!\!/_{\mathcal{E}} |D\!\!\!/_{\mathcal{E}}|^{-(s+1)} = \Gamma\left(\frac{s+1}{2}\right)^{-1} \int_0^\infty t^{\frac{s-1}{2}} D\!\!\!/_{\mathcal{E}} \mathrm{e}^{-tD\!\!\!/_{\mathcal{E}}^2}\, \mathrm{d}t. \tag{1.2}$$

For $t > 0$ let $h_t(x, y) \in C^\infty(M, |\Lambda|(M) \otimes \operatorname{End}\mathcal{E})$ be the kernel of $D\!\!\!/_{\mathcal{E}} \mathrm{e}^{-tD\!\!\!/_{\mathcal{E}}^2}$, where $|\Lambda|(M)$ denotes the bundle of densities on M. Since by standard heat kernel asymptotics we have $\operatorname{tr}_{\mathcal{E}} h_t(x, x) = \mathrm{O}(t^{-n/2})$ as $t \to 0^+$ (see Theorem 2.10 ahead), for $\Re s > n - 1$ we get

$$\eta(D\!\!\!/_{\mathcal{E}}; s)(x) = \Gamma\left(\frac{s+1}{2}\right)^{-1} \int_0^\infty t^{\frac{s-1}{2}} \operatorname{tr}_{\mathcal{E}} h_t(x, x)\, \mathrm{d}t. \tag{1.3}$$

Then Bismut–Freed ([10, Thm. 2.4]) proved that in the C^0-topology we have

$$\operatorname{tr}_{\mathcal{E}} h_t(x, x) = \mathrm{O}(\sqrt{t}) \qquad \text{as } t \to 0^+. \tag{1.4}$$

It thus follows that the local eta function $\eta(D\!\!\!/_{\mathcal{E}}; s)(x)$ is actually holomorphic for $\Re s > -2$. In particular, we have the formula,

$$\eta(D\!\!\!/_{\mathcal{E}}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr} D\!\!\!/_{\mathcal{E}} \mathrm{e}^{-tD\!\!\!/_{\mathcal{E}}^2}\, \mathrm{d}t, \tag{1.5}$$

which, for instance, plays a crucial role in the study of the adiabatic limit of the eta invariant of a Dirac operator (see [9,10]). Incidentally, Bismut–Freed's asymptotics (1.4), which is a purely analytic statement, implies the global regularity of the eta invariant of any selfadjoint elliptic ΨDO, as well as the aforementioned generalizations of Wodzicki.

Now, the standard proof of the asymptotics (1.4) is essentially based on a reduction to the local index theorem of Patodi, Gilkey, Atiyah–Patodi–Singer ([1,16]; see also [13]), which provides us with a heat kernel proof of the index theorem of Atiyah–Singer [2,3] for Dirac operators. In the original proof of (1.4) in [10] the reduction is done by introducing an extra Grassmanian variable z, $z^2 = 0$, and in [18, Sect. 8.3] by means of a suspension argument. Moreover, in [10] the authors refer to the results of [17] to justify the differentiability of the heat kernel asymptotics.

On the other hand, in [20] the approach to the heat kernel asymptotics of [17] was combined with general considerations on Getzler's order of Volterra ΨDO's to produce a new short proof of the local index theorem which holds in many other settings. Furthermore, the arguments used in this proof have other applications such as the computation of the CM cyclic cocycle of [12] for Dirac operators.

The aim of this paper is to show that we can get a direct proof of Bismut–Freed's asymptotics by using the approach of [20]. In fact, once the background from [17,20] is set-up, the proof becomes extremely simple since it is deduced by combining the considerations on Getzler orders of [20] with the observation that the subleading terms (in the Getzler order sense) of the various asymptotics at stake vanish at first order.

As in [20] the approach of this paper is rather general and should therefore hold in various other settings. Moreover, it is believed that this approach could also be useful to study adiabatic-type limits of eta invariants. For instance, it has been shown by Rumin [24] that, under the so-called subriemannian limit, the differential form spectrum of the de Rham complex on a contact manifold converges to that of the contact complex [23], so it could be fruitful to use the approach of this paper to further study the behavior under the subriemannian limit of the eta invariant on a contact manifold.

This paper is organized as follows. In Section 2 we recall Greiner's approach to the heat kernel asymptotics. In Section 3, after having recalled the background of [20], we prove Bismut–Freed's asymptotics.

2. Greiner's approach of the heat kernel asymptotics

In this section we recall the approach of the heat kernel asymptotics of Greiner [17]. Here we let M^n be a compact Riemannian manifold, let $\mathcal{E}$ be a Hermitian vector bundle over M and we consider a selfadjoint second order elliptic differential operator $\Delta : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$ with principal symbol $a_2(x, \xi) > 0$. Then Δ is bounded from below on $L^2(M, \mathcal{E})$ and by standard functional calculus we can define $\mathrm{e}^{-t\Delta}$, $t \geq 0$, as a selfadjoint bounded operator on $L^2(M, \mathcal{E})$. In fact, $\mathrm{e}^{-t\Delta}$ is smoothing for $t > 0$, so its Schwartz kernel $k_t(x, y)$ is in $C^\infty(M, \mathcal{E}) \hat{\otimes} C^\infty(M, \mathcal{E}^* \otimes |\Lambda|(M))$, where $|\Lambda|(M)$ denotes the bundle of densities on M.

Recall that the heat semigroup allows us to invert the heat equation, in the sense that the operator,

$$Q_0 u(x, t) = \int_0^\infty \mathrm{e}^{-s\Delta} u(x, t - s)\, \mathrm{d}s, \qquad u \in C_c^\infty(M \times \mathbb{R}, \mathcal{E}), \tag{2.1}$$

maps continuously into $C^0(\mathbb{R}, L^2(M, \mathcal{E})) \subset \mathcal{D}'(M \times \mathbb{R}, \mathcal{E})$ and satisfies

$$(\Delta + \partial_t) Q_0 u = Q_0 (\Delta + \partial_t) u = u \qquad \forall u \in C_c^\infty(M \times \mathbb{R}, \mathcal{E}). \tag{2.2}$$

Notice that the operator Q_0 has the *Volterra property* in the sense of [19]: it is translation invariant and satisfies the causality principle, i.e., Q has a distribution kernel of the form $K_{Q_0}(x, y, t - s)$ where $K_{Q_0}(x, y, t)$ vanishes on the region $t < 0$. In fact, we have

$$K_{Q_0}(x, y, t) = \begin{cases} k_t(x, y), & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{2.3}$$

The above equalities lead us to use pseudodifferential techniques to study the heat kernel $k_t(x, y)$. The idea, which goes back to Hadamard, is to consider a class of ΨDO's, the Volterra ΨDO's of Greiner [17] and Piriou [19], taking into account:

(i) The aforementioned Volterra property;
(ii) The parabolic homogeneity of the heat operator $\Delta + \partial_t$, i.e., the homogeneity with respect to the dilations,

$$\lambda(\xi, \tau) = (\lambda\xi, \lambda^2\tau), \qquad (\xi, \tau) \in \mathbb{R}^{n+1}, \quad \lambda \neq 0. \tag{2.4}$$

In the sequel for $g \in \mathcal{S}'(\mathbb{R}^{n+1})$ and $\lambda \neq 0$ we let g_λ be the tempered distribution defined by

$$\langle g_\lambda(\xi,\tau)u(\xi,\tau)\rangle = |\lambda|^{-(n+2)}\langle g(\xi,\tau)u(\lambda^{-1}\xi,\lambda^{-2}\tau)\rangle, \quad u \in \mathcal{S}(\mathbb{R}^{n+1}). \tag{2.5}$$

We then say that g is *parabolic homogeneous* of degree m, $m \in \mathbb{Z}$, when we have $g_\lambda = \lambda^m g$ for any $\lambda \neq 0$.

Let $\mathbb{C}_-$ denote the complex halfplane $\{\Im\tau > 0\}$ with closure $\overline{\mathbb{C}_-} \subset \mathbb{C}$. Then:

Lemma 2.1. ([6, Prop. 1.9]) *Let $q(\xi,\tau) \in C^\infty((\mathbb{R}^n \times \mathbb{R}) \setminus 0)$ be a parabolic homogeneous symbol of degree m such that:*

(i) *q extends to a continuous function on $(\mathbb{R}^n \times \overline{\mathbb{C}_-}) \setminus 0$ in such way to be holomorphic in the last variable when the latter is restricted to $\mathbb{C}_-$.*

Then there is a unique $g \in \mathcal{S}'(\mathbb{R}^{n+1})$ agreeing with q on $\mathbb{R}^{n+1} \setminus 0$ so that:

(ii) *g is parabolic homogeneous of degree m;*
(iii) *The inverse Fourier transform $\check{g}(x,t)$ vanishes for $t < 0$.*

Let U be an open subset of $\mathbb{R}^n$. We define Volterra ΨDO's on $U \times \mathbb{R}$ as follows.

Definition 2.2. $S_{\mathrm{v}}^m(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists of smooth functions $q(x,\xi,\tau)$ on $U \times \mathbb{R}^n \times \mathbb{R}$ with an asymptotic expansion $q \sim \sum_{j\geq 0} q_{m-j}$, where:

- The symbol $q_l(x,\xi,\tau) \in C^\infty(U \times [(\mathbb{R}^n \times \mathbb{R}) \setminus 0])$ is parabolic homogeneous of degree l and satisfies the property (i) in Lemma 2.1 with respect to the variables ξ and τ;
- The sign $\sim$ means that, for any integer N and any compact $K \subset U$, there is a constant $C_{NK\alpha\beta k} > 0$ such that, for $x \in K$ and $|\xi| + |\tau|^{\frac{1}{2}} > 1$, we have

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k (q - \sum_{j<N} q_{m-j})(x,\xi,\tau)| \leq C_{NK\alpha\beta k}(|\xi| + |\tau|^{1/2})^{m-N-|\beta|-2k}. \tag{2.6}$$

Given a symbol $q \in S_{\mathrm{v}}^m(U \times \mathbb{R}^{n+1})$ we quantize it by associating to it the operator $Q(x,D_x,D_t) : C_c^\infty(U_x \times \mathbb{R}_t) \to C^\infty(U_x \times \mathbb{R}_t)$ such that, for any $u \in C_c^\infty(U_x \times \mathbb{R}_t)$, we have

$$Q(x,D_x,D_t)u(x,t) = (2\pi)^{-(n+1)} \int e^{i(x.\xi+t.\tau)} q(x,\xi,\tau)\check{u}(\xi,\tau)\, d\xi\, d\tau. \tag{2.7}$$

This is a continuous operator from $C_c^\infty(U_x \times \mathbb{R}_t)$ to $C^\infty(U_x \times \mathbb{R}_t)$ satisfying the Volterra property and with distribution kernel $\check{q}_{(\xi,\tau)\to(y,t)}(x,y,t-s)$.

Definition 2.3. $\Psi_{\mathrm{v}}^m(U \times \mathbb{R})$, $m \in \mathbb{Z}$, consists of continuous operators Q from $C_c^\infty(U_x \times \mathbb{R}_t)$ to $C^\infty(U_x \times \mathbb{R}_t)$ such that:

(i) Q has the Volterra property;
(ii) Q is of the form $Q = q(x,D_x,D_t) + R$ for some symbol q in $S_{\mathrm{v}}^m(U \times \mathbb{R})$ and some smoothing operator R.

In the sequel if Q is a Volterra ΨDO we let $K_Q(x,y,t-s)$ denote its distribution kernel, so that the distribution $K_Q(x,y,t)$ vanishes for $t < 0$.

Examples of Volterra ΨDO includes differential operators, as well as homogeneous operators below.

Definition 2.4. Let $q_m(x,\xi,\tau) \in C^\infty(U \times (\mathbb{R}^{n+1} \setminus 0))$ be a homogeneous Volterra symbol of order m and let $g_m \in C^\infty(U)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n+1})$ denote its unique homogeneous extension given by Lemma 2.1. Then:

- $\check{q}_m(x,y,t)$ denotes the inverse Fourier transform $(g_m)^\vee_{(\xi,\tau)\to(y,t)}(x,y,t)$;
- $q_m(x,D_x,D_t)$ denotes the operator with kernel $\check{q}_m(x,y-x,s-t)$.

Remark 2.5. The above definition makes sense since it follows from the proof of Lemma 2.1 in [6] that the extension process of Lemma 2.1 applied to each symbol $q_m(x,.,.)$, $x \in U$, is smooth with respect to x, so really gives rise to an element of $C^\infty(U)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n+1})$.

Proposition 2.6. ([17,19]) *The following properties hold.*

1) Composition. *Let $Q_j \in \Psi_{\mathrm{v}}^{m_j}(U \times \mathbb{R})$, $j=1,2$, have symbol q_j and suppose that Q_1 or Q_2 is properly supported. Then Q_1Q_2 belongs to $\Psi_{\mathrm{v}}^{m_1+m_2}(U\times\mathbb{R})$ and has symbol $q_1\#q_2 \sim \sum 1/\alpha!\partial_\xi^\alpha q_1 D_x^\alpha q_2$.*
2) Parametrices. *An operator $Q \in \Psi_{\mathrm{v}}^m(U\times\mathbb{R})$ has a parametrix in $\Psi_{\mathrm{v}}^{-m}(U\times\mathbb{R})$ if, and only if, its principal symbol is nowhere vanishing on $U \times [(\mathbb{R}^n \times \overline{\mathbb{C}_-} \setminus 0)]$.*
3) Invariance. *Let $\phi : U \to V$ be a diffeomorphism onto another open subset V of $\mathbb{R}^n$ and let Q be a Volterra ΨDO on $U\times\mathbb{R}$ of order m. Then $Q = (\phi \oplus \mathrm{id}_{\mathbb{R}})_* Q$ is a Volterra ΨDO on $V \times \mathbb{R}$ of order m.*

Remark 2.7. The proofs of the first and third properties above follow along the same lines as that of the corresponding proofs for standard ΨDO's. For the second one we need the Volterra calculus to be asymptotically complete. The latter fact cannot, however, be reached by means of the standard proof for classical ΨDO's, because the cut-off arguments therein are not valid valid anymore with analytic symbols. Nevertheless, simple proofs can be found in [21].

The key property of Volterra ΨDO's which allows us to derive a differentiable heat kernel asymptotics is the following.

Lemma 2.8. ([17, Chap. I]) *Let $Q \in \Psi_{\mathrm{v}}^m(U\times\mathbb{R})$ have symbol $q \sim \sum_{j\geq 0} q_{m-j}$. Then the following asymptotics holds in $C^\infty(U)$,*

$$K_Q(x,x,t) \sim_{t\to 0^+} t^{-(n/2+[m/2]+1)} \sum_{l\geq 0} t^l \check{q}_{2[m/2]-2l}(x,0,1), \tag{2.8}$$

where the notation $\check{q}_k$ has the same meaning as in Definition 2.4.

Proof. As the Fourier transform relates the decay at infinity to the behavior at the origin of the Fourier transform, the distribution $\check{q} - \sum_{j\leq J} \check{q}_{m-j}$ is in $C^N(U_x \times \mathbb{R}^n_y \times \mathbb{R}_t)$ as soon as J is large enough. Since $Q - q(x,D_x,D_t)$ is smoothing, we see that $R_J(x,t) = K_Q(x,x,t) - \sum_{j\leq J} \check{q}_{m-j}(x,0,t)$ is of class C^N. As $R_J(x,y,t) = 0$ for $t<0$ we get $\partial_t^l R_J(x,0) = 0$ for $l = 0,1,\ldots,N$, so that $R_J(.,t)$ is a $\mathrm{O}(t^N)$ in $C^N(U)$ as $t \to 0^+$. It then follows that in $C^\infty(U)$ we have

$$K_Q(x,x,t) \sim \sum_{j\geq 0} t^{-(n/2+1)} \check{q}_{m-j}(x,0,t) \qquad \text{as } t \to 0^+. \tag{2.9}$$

Now, using (2.5) we see that for $\lambda \neq 0$ we have

$$(\check{q}_{m-j})_\lambda = |\lambda|^{-(n+2)}(q_{m-j,\lambda^{-1}})^\vee = |\lambda|^{-(n+2)}\lambda^{j-m}\check{q}_{m-j}. \tag{2.10}$$

Setting $\lambda = \sqrt{t}$ with $t > 0$ then gives $\check{q}_{m-j}(x, 0, t) = t^{((j-n-m)/2)+1}\check{q}_{m-j}(x, 0, 1)$. Thus,

$$K_Q(x, x, t) \sim \sum_{j\geq 0} t^{((j-n-m)/2)-1}\check{q}_{m-j}(x, 0, 1) \qquad \text{as } t \to 0^+. \tag{2.11}$$

On the other hand, setting $\lambda = -1$ in (2.10) shows that when $m - j$ is odd we have $\check{q}_{m-j}(x, 0, 1) = -\check{q}_{m-j}(x, 0, 1) = 0$. Therefore, in (2.11) only the symbols of even degree contributes to the asymptotics, i.e., we get (2.8). □

The invariance property in Proposition 2.6 allows us to define Volterra ΨDO's on $M \times \mathbb{R}$ acting on the sections of the vector bundle $\mathcal{E}$. All the preceding properties hold *verbatim* in this context. In particular the heat operator $\Delta + \partial_t$ admits a parametrix Q in $\Psi_{\mathrm{v}}^{-2}(M \times \mathbb{R}, \mathcal{E})$.

In fact, comparing the operator (2.1) with any Volterra ΨDO-parametrix for $\Delta + \partial_t$ allows us to prove:

Theorem 2.9. ([17,19]; see also [6, pp. 362–363]) *The differential operator $\Delta + \partial_t$ has an inverse Q_0 in $\Psi_{\mathrm{v}}^{-2}(M \times \mathbb{R}, \mathcal{E})$, so that $Q_0(\Delta + \partial_t) = (\Delta + \partial_t)Q_0 = 1$.*

We can now get the heat kernel asymptotics in the differentiable version below.

Theorem 2.10. ([17, Thm. 1.6.1]) *Let $P : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$ be a differential operator of order m and let $h_t(x, y)$ denote the distribution kernel of $P\mathrm{e}^{-t\Delta}$. Then in $C^\infty(M, |\Lambda|(M) \otimes \operatorname{End}\mathcal{E})$ as $t \to 0^+$ we have*

$$h_t(x, x) \sim t^{-[m/2]-n/2} \sum_{l\geq 0} t^l b_l(P, \Delta)(x), \quad b_l(P, \Delta)(x) = \check{q}_{2[\frac{m}{2}]-2-2l}(x, 0, 1), \tag{2.12}$$

where the equality on the right-hand side gives a formula for computing the densities $b_l(P, \Delta)(x)$'s in local trivializing coordinates using the symbol $q \sim \sum_{j\geq 0} q_{m-2-j}$ of $Q = P(\Delta + \partial_t)^{-1}$.

Proof. As $h_t(x, y) = P_x k_t(x, y) = P_x K_{(\Delta+\partial_t)^{-1}}(x, y, t) = K_{P(\Delta+\partial_t)^{-1}}(x, y, t)$ the result follows by applying Lemma 2.8 to $P(\Delta + \partial_t)^{-1}$. □

3. Proof of Bismut–Freed's asymptotics

In this section we shall show that implementing the rescaling of Getzler [13] into the Volterra calculus allows us to get a new proof of Bismut–Freed's asymptotics (1.4).

Let (M^n, g) be a compact Riemannian manifold of odd dimension and let $\mathrm{Cl}(M)$ denote its Clifford bundle, so that the fiber $\mathrm{Cl}_x(M)$ at $x \in M$ is the complex algebra generated by 1 and T_x^*M with relations,

$$\xi \cdot \eta + \eta \cdot \xi = -2\langle \xi, \eta\rangle, \qquad \xi, \eta \in T_x^*M. \tag{3.1}$$

For $\xi \in \Lambda T_{\mathbb{C}}^*M$ and $l = 0, \ldots, n$ we let $\xi^{(l)}$ denote component of ξ in $\Lambda^l T_{\mathbb{C}}^*M$. Then the quantization map $c : \Lambda T_{\mathbb{C}}^*M \to \mathrm{Cl}(M)$ and its inverse $\sigma = c^{-1}$ are such that for ξ and η in $\Lambda T_{\mathbb{C}}^*M$ we have

$$\sigma(c(\xi^{(i)})c(\eta^{(j)})) = \xi^{(i)} \wedge \eta^{(j)} \mod \Lambda^{i+j-2}T_{\mathbb{C}}^*M. \tag{3.2}$$

Now, let $\mathcal{E}$ be a Clifford bundle over M equipped with a unitary Clifford connection $\nabla^{\mathcal{E}}$ and let $D\!\!\!/_{\mathcal{E}}$ be the associated Dirac operator,

$$D\!\!\!/_{\mathcal{E}} : C^\infty(M, \mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} C^\infty(M, T^*M \otimes \mathcal{E}) \xrightarrow{c} C^\infty(M, \mathcal{E}), \tag{3.3}$$

where c denotes the Clifford action of ΛT^*M on $\mathcal{E}$. Recall that by the Lichnerowicz's formula (see, e.g., [7, Thm. 3.52]) we have

$$D\!\!\!/_{\mathcal{E}}^2 = (\nabla_i^{\mathcal{E}})^* \nabla_i^{\mathcal{E}} + \mathcal{F}^{\mathcal{E}/\$} + \frac{\kappa^M}{4}, \tag{3.4}$$

where κ^M denotes the scalar curvature of M and $\mathcal{F}^{\mathcal{E}/\$}$ is the twisted curvature $F^{\mathcal{E}/\$}$ acting by Clifford multiplication on $\mathcal{E}$.

For $t > 0$ let $h_t(x, y)$ denote the kernel of $D\!\!\!/_{\mathcal{E}} e^{-tD\!\!\!/_{\mathcal{E}}^2}$. The aim of this section is to prove Bismut–Freed's asymptotics in the version below.

Theorem 3.1. ([10, Thm. 2.4]) *In $C^\infty(M, |\Lambda|(M) \otimes \mathrm{End}\mathcal{E})$ we have*

$$\mathrm{tr}_{\mathcal{E}} h_t(x, x) = \mathrm{O}(\sqrt{t}) \qquad \text{as } t \to 0^+. \tag{3.5}$$

First, as by Theorem 2.10 we already have an asymptotics in $C^\infty(M, |\Lambda|(M))$ for $\mathrm{tr}_{\mathcal{E}} h_t(x, x)$ as $t \to 0^+$ it is enough to prove (3.5) at a point $x_0 \in M$. In fact, proving (3.5) at x_0 is a purely local issue since in local trivializing coordinates near x_0 the coefficients of the asymptotics for $\mathrm{tr}_{\mathcal{E}} h_t(x, x)$ depend only the homogeneous components of the symbol of $D\!\!\!/_{\mathcal{E}}(D\!\!\!/_{\mathcal{E}}^2 + \partial_t)^{-1}$. In particular, if in local trivializing coordinates near x_0 we let $Q_{\mathcal{E}}$ be a Volterra ΨDO parametrix for $D\!\!\!/_{\mathcal{E}}^2 + \partial_t$ then we have

$$h_t(x_0, x_0) = K_{D\!\!\!/_{\mathcal{E}} Q_{\mathcal{E}}}(x_0, x_0, t) + \mathrm{O}(t^\infty) \qquad \text{as } t \to 0^+. \tag{3.6}$$

As usual it will be convenient to use normal coordinates centered at x_0 and trivializations of TM and $\mathcal{E}$ by means of synchronous orthogonal frames, assuming that the synchronous tangent frame $e_1, \ldots, e_n$ is such that $e_j = \partial_j$ at $x = 0$. This allows us to:

- Replace $D\!\!\!/_{\mathcal{E}}$ by a Dirac operator $D\!\!\!/_{\mathcal{E}}^p$ on $\mathbb{R}^n$ acting on a trivial twisted bundle with fiber $\$_n \otimes \mathbb{C}^p$, where $\$_n$ denotes the spinor space of $\mathbb{R}^n$.
- Have a metric g and coefficients $\omega_{ikl} = \langle \nabla_i^{LC} e_k e_l \rangle$ of the Levi-Civita connection with behaviors near $x = 0$ of the form

$$g_{ij}(x) = \delta_{ij} + \mathrm{O}(|x|^2), \qquad \omega_{ikl}(x) = -\tfrac{1}{2} R^M_{ijkl}(0) x^j + \mathrm{O}(|x|^2), \tag{3.7}$$

where $R^M_{ijkl}(0) = \langle R^M(0)(\partial_i, \partial_j)\partial_k \partial_l \rangle$.

Second, let $\mathrm{Cl}(n)$ and $\Lambda(n)$ respectively denote the Clifford algebra and the (complexified) exterior algebra of $\mathbb{R}^n$. Since the dimension n is odd if we regard $c(\mathrm{d}x^{i_1}) \cdots c(\mathrm{d}x^{i_k})$, $i_1 < \cdots < i_k$, as an endomorphism of $\$_n$ then, as observed in [10, p. 107], we have

$$\mathrm{tr}_{\$_n} c(\mathrm{d}x^{i_1}) \cdots c(\mathrm{d}x^{i_k}) = \begin{cases} 2^{[n/2]} & \text{if } k = 0, \\ 0 & \text{if } 0 < k < n, \\ (-i)^{[n/2]+1} 2^{[n/2]} & \text{if } k = n. \end{cases} \tag{3.8}$$

Next, the Dirac operator $D\!\!\!/_{\mathbb{C}^p}$ is obtained by composing the action of $\mathrm{Cl}(n)$ on $\$_n$ with the Clifford Dirac operator $D\!\!\!/ = c(\mathrm{d}x^i)\nabla_i$ with coefficients in $\mathrm{Cl}^{\mathrm{odd}}(n)$. Since $\mathrm{Cl}^{\mathrm{ev}}(n)$ acts on itself by multiplication, any Volterra ΨDO-parametrix Q for $D\!\!\!/^2 + \partial_t$ has coefficients in $\mathrm{Cl}^{\mathrm{ev}}(n)$ up to a smoothing operator. By composing $\not{D}Q$ with the Clifford action we get the operator $\not{D}_{\mathbb{C}^p}\tilde{Q}$, where $\tilde{Q}$ is the Volterra ΨDO-parametrix of $\not{D}^2_{\mathbb{C}^p} + \partial_t$ corresponding to Q. Since $\not{D}Q$ has coefficients in $\mathrm{Cl}^{\mathrm{odd}}(n)$ up to a smoothing operator, using (3.6) and (3.8) we see that as $t \to 0^+$ we have

$$\begin{aligned}\mathrm{tr}_{\mathcal{E}} h_t(0,0) &= \mathrm{tr}_{\not{S}_n \otimes \mathbb{C}^p} K_{\not{D}^2_{\mathbb{C}^p}}(0,0,t) + \mathrm{O}(t^\infty) \\ &= (-i)^{[\frac{n}{2}]+1} 2^{[\frac{n}{2}]} (\sigma \otimes \mathrm{tr}_{\mathbb{C}^p})[K_{\not{D}Q}(0,0,t)]^{(n)} + \mathrm{O}(t^\infty).\end{aligned} \tag{3.9}$$

Therefore, we are reduced to study the small time behavior of $(\sigma \otimes \mathrm{tr}_{\mathbb{C}^p})[K_{\not{D}Q}(0,0,t)]^{(n)}$.

Now, the Getzler's rescaling [13] aims to assign the degrees,

$$\deg \partial_j = \deg c(\,\mathrm{d}x^j) = 1, \quad \deg \partial_t = 2, \quad \deg x^j = -1, \tag{3.10}$$

while $\deg B = 0$ for any $B \in M_p(\mathbb{C})$. As pointed out in [20] this allows us to define a filtration on Volterra ΨDO's with coefficients in $\mathrm{Cl}(n)$ as follows.

Let $Q \in \Psi^*_{\mathrm{v}}(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p) \otimes \mathrm{Cl}(n)$ have symbol $q(x,\xi,\tau) \sim \sum_{k \le m'} q_k(x,\xi,\tau)$. Then taking components in each subspace $\Lambda^j(n)$ and then using Taylor expansions at $x = 0$ gives formal expansions

$$\sigma[q(x,\xi,\tau)] \sim \sum_{j,k} \sigma[q_k(x,\xi,\tau)]^{(j)} \sim \sum_{j,k,\alpha} \frac{x^\alpha}{\alpha!} \sigma[\partial_x^\alpha q_k(0,\xi,\tau)]^{(j)}, \tag{3.11}$$

where the last expansion is taken with respect to the filtration of $S^m_{\mathrm{v}}(\mathbb{R}^n \times \mathbb{R}^{n+1}, \mathbb{C}^p) \otimes \Lambda(n)$ by the subspaces $x^\alpha S^k_{\mathrm{v}}(\mathbb{R}^n \times \mathbb{R}^{n+1}, \mathbb{C}^p) \otimes \Lambda^j(n)$.

Motivated by (3.10) we shall say that the symbol $\frac{x^\alpha}{\alpha!} \partial_x^\alpha \sigma[q_k(0,\xi,\tau)]^{(j)}$ is Getzler homogeneous of degree $k + j - |\alpha|$. Thus, we have an asymptotic expansion,

$$\sigma[q(x,\xi,\tau)] \sim \sum_{j \ge 0} q_{(m-j)}(x,\xi,\tau), \qquad q_{(m)} \neq 0, \tag{3.12}$$

where the symbol $q_{(m-j)}$ is Getzler homogeneous of degree $m - j$.

We shall call m the *Getzler order* of Q. Moreover, extending the Definition 2.4 to homogeneous symbols with coefficients in $(\mathrm{End}\,\mathbb{C}^p) \otimes \Lambda(n)$, we define the *model operator* of Q as the element of $\Psi^*_{\mathrm{v}}(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p) \otimes \Lambda(n)$ given by

$$Q_{(m)} := q_{(m)}(x, D_x, D_t). \tag{3.13}$$

In the sequel we will write $\mathrm{O_G}(m)$ to denote a Volterra ΨDO of Getzler order $\le m$ and we will write $x\mathrm{O_G}(m)$ to denote an element of $\Psi^*_{\mathrm{v}}(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p) \otimes \mathrm{Cl}(n)$ of the form $x^i Q_i$ with Q_i of Getzler order $\le m$.

Let $A = A_i\,\mathrm{d}x^i$ be the connection one-form on $\mathbb{C}^p$. Then by (3.7) the covariant derivative $\nabla_i = \partial_i + \frac{1}{4}\omega_{ikl}(x)c(e^k)c(e^l) + A_i$ on $\mathrm{Cl}(n) \otimes \mathbb{C}^p$ has Getzler order 1 and model operator,

$$\nabla_{i(1)} = \partial_i - \tfrac{1}{4} R^M_{ij}(0) x^j, \qquad R^M_{ij}(0) = R^M_{ijkl}(0)\,\mathrm{d}x^k \wedge \mathrm{d}x^l. \tag{3.14}$$

This implies that $\not{D} = c(\mathrm{d}x^i)\nabla_i$ has Getzler order 2 and can be expanded as

$$\not{D} = c(\not{D}_{(2)}) + x\mathrm{O_G}(2), \qquad \not{D}_{(2)} = \varepsilon(\,\mathrm{d}x^i)\nabla_{i(1)}, \tag{3.15}$$

where $\varepsilon(\,\mathrm{d}x^i)$ denotes the exterior multiplication by $\mathrm{d}x^i$.

The interest to introduce Getzler orders stems from the two lemmas below.

Lemma 3.2. ([20, Lem. 3]) *Let $Q \in \Psi_{\mathrm{v}}^{*}(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p) \otimes \mathrm{Cl}(n)$ have Getzler order m and model operator $Q_{(m)}$.*

(i) *If m is odd, then as $t \to 0^+$ we have*

$$\sigma[K_Q(0,0,t)]^{(n)} = t^{-(\frac{m}{2}+1)} K_{Q_{(m)}}(0,0,1)^{(n)} + \mathrm{O}(t^{-\frac{m}{2}}). \tag{3.16}$$

(ii) *If m is even, then $K_{Q_{(m)}}(0,0,t)^{(n)} = 0$ and as $t \to 0^+$ we have*

$$\sigma[K_Q(0,0,t)]^{(n)} = \mathrm{O}(t^{-\frac{m+1}{2}}). \tag{3.17}$$

Proof. Let $q(x,\xi,\tau) \sim \sum_{j \leq m'} q_j(x,\xi,\tau)$ be the symbol of Q and let $q_{(m)}(x,\xi,\tau)$ be its principal Getzler homogeneous symbol. Then by Lemma 2.8 as $t \to 0^+$ we have $\sigma[K_Q(0,0,t)]^{(n)} \sim \sum_{j \leq m'} t^{-\frac{n+2-j}{2}} \sigma[\check{q}_j(0,0,1)]^{(n)}$. Notice that $\sigma[q_j(0,\xi,\tau)]^{(n)}$ is Getzler homogeneous of degree $j+n$, so it must be zero if $j+n > m$, since otherwise Q would have Getzler order $> m$. Therefore, we get:

$$\sigma[K_Q(0,0,t)]^{(n)} = t^{-(\frac{m}{2}+1)} \sigma[\check{q}_{m-n}(0,0,1)]^{(n)} + t^{-\frac{m+1}{2}} \sigma[\check{q}_{m-n-1}(0,0,1)]^{(n)} + \mathrm{O}(t^{-\frac{m}{2}}). \tag{3.18}$$

On the other hand, in view of asymptotics (3.11) and (3.12) the symbol $q_{(m)}(0,\xi,\tau)^{(n)}$ is equal to $\sum_{j+n-|\alpha|=m} (\frac{x^\alpha}{\alpha!} \partial_x^\alpha \sigma[q_j(0,\xi,\tau)]^{(n)})_{x=0} = \sigma[q_{m-n}(0,\xi,\tau)]^{(n)}$. Thus,

$$\sigma[\check{q}_{m-n}(0,0,1)]^{(n)} = \check{q}_{(m)}(0,0,1)^{(n)} = K_{Q_{(m)}}(0,0,1)^{(n)}. \tag{3.19}$$

Finally, as pointed out in the proof of Lemma 2.8 we have $\check{q}_j(0,0,1) = 0$ when j is odd. As n is odd we see that if m odd we have $\sigma[\check{q}_{m-n-1}(0,0,1)]^{(n)} = 0$ and so (3.16) holds, while if m even then $K_{Q_{(m)}}(0,0,1)^{(n)} = \sigma[\check{q}_{m-n}(0,0,1)]^{(n)} = 0$ and we obtain (3.17). □

Lemma 3.3. *For $j = 1,2$ let $Q_j \in \Psi_{\mathrm{v}}^{*}(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p) \otimes \mathrm{Cl}(n)$ have Getzler order m_j and model operator $Q_{(m_j)}$, and assume that Q_1 or Q_2 is properly supported. Then in $\Psi_{\mathrm{v}}^{*}(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p) \otimes \mathrm{Cl}(n)$ we have*

$$Q_1 Q_2 = c[Q_{(m_1)} Q_{(m_2)}] + \mathrm{O}_G(m_1 + m_2 - 1), \tag{3.20}$$

$$c(Q_{(m_1)}) c(Q_{(m_2)}) = c[Q_{(m_1)} Q_{(m_2)}] + \mathrm{O}_G(m_1 + m_2 - 2). \tag{3.21}$$

Proof. The equality (3.20) is the content of Lemma 4 of [20], so we need only to prove (3.21). For for a subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ with $i_1 < \ldots < i_k$ we let $\mathrm{d}x^I = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$. Since the forms $\mathrm{d}x^I$ gives rise to a linear basis of $\Lambda(n)$ we can write $Q_{(m_j)}$, $j = 1,2$, in the form $Q_{(m_j)} = \sum Q_{j,I} \, \mathrm{d}x^I$ with $Q_{j,I}$ in $\Psi_{\mathrm{v}}^{*}(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p)$ of Getzler order $m_j - |I|$. Then we have

$$c(Q_{(m_1)}) c(Q_{(m_2)}) = \sum_{I_1, I_2} Q_{1,I_1} Q_{2,I_2} c(\mathrm{d}x^{I_1}) c(\mathrm{d}x^{I_2}). \tag{3.22}$$

Thanks to (3.2) we have $c(\mathrm{d}x^{I_1}) c(\mathrm{d}x^{I_2}) = c(\mathrm{d}x^{I_1}) \wedge \mathrm{d}x^{I_2}) + \mathrm{O}_G(|I_1| + |I_2| - 2)$, so we obtain

$$\begin{aligned} c(Q_{(m_1)}) c(Q_{(m_2)}) &= \sum_{I_1, I_2} Q_{1,I_1} Q_{2,I_2} (c(\mathrm{d}x^{I_1}) \wedge \mathrm{d}x^{I_2}) + \mathrm{O}_G(|I_1| + |I_2| - 2) \\ &= c[Q_{(m_1)} Q_{(m_2)}] + \mathrm{O}_G(m_1 + m_2 - 2). \end{aligned} \tag{3.23}$$

The proof is thus achieved. □

Now, by the Lichnerowicz's formula (3.4) we have

$$\not{D}^2 = -g^{ij}(\nabla_i\nabla_j - \Gamma_{ij}^k\nabla_k) + \frac{1}{2}c(\mathrm{e}^{\mathrm{i}})c(\mathrm{e}^{\mathrm{j}})F(e_i, e_j) + \frac{\kappa}{4}, \tag{3.24}$$

where the Γ_{ij}^k's are the Christoffel symbols of the metric g. Observe that:

- Thanks to (3.7) we have $g_{ij}(x) = \delta_{ij} + \mathrm{O}(|x|^2) = \delta_{ij} + \mathrm{O}_G(-2)$;
- Using (3.20) and (3.14), as well as the fact that $\Gamma_{ij}^k(x) = \mathrm{O}(|x|) = \mathrm{O}_G(-1)$, we get $\nabla_i\nabla_j = c(\nabla_{i(1)}\nabla_{j(1)}) + x\mathrm{O}_G(2) + \mathrm{O}_G(0)$;
- We have $F^{\mathcal{E}}(e^i, e^j) = F^{\mathcal{E}}(\partial_i, \partial_j)(0) + \mathrm{O}(|x|) = F^{\mathcal{E}}(\partial_i, \partial_j)(0) + x\mathrm{O}_G(0)$;
- By (3.2) we have $c(\mathrm{e}^{\mathrm{i}})c(\mathrm{e}^{\mathrm{j}}) = c(\mathrm{e}^{\mathrm{i}} \wedge \mathrm{e}^{\mathrm{j}}) + \mathrm{O}_G(0) = c(\,\mathrm{d}x^i \wedge \mathrm{d}x^j) + x\mathrm{O}_G(2) + \mathrm{O}_G(0)$.

Moreover, as x^i commutes with x^j and $c(\,\mathrm{d}x^j)$ and commutes with Volterra ΨDO's of order m modulo those of order $\leq m-1$, we see that the commutation with x^i preserves the Getzler order. In particular, if $Q \in \Psi_{\mathrm{v}}^*(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p) \otimes \mathrm{Cl}(n)$ has Getzler order $\leq m$, then it follows from the equality $Qx^i = x^iQ + [Q, x^i]$ that Qx^i is of the form $x\mathrm{O}_G(m) + \mathrm{O}_G(m)$.

Bearing all this in mind we obtain:

$$\begin{aligned}\not{D}^2 &= (\delta_{ij} + \mathrm{O}_G(-2))(c(\nabla_{i(1)}\nabla_{j(1)}) + x\mathrm{O}_G(2) + \mathrm{O}_G(0)) \\ &\quad + \tfrac{1}{2}(F^{\mathcal{E}}(\partial_k, \partial_l)(0) + x\mathrm{O}_G(0))(c(\,\mathrm{d}x^i \wedge \mathrm{d}x^j) + x\mathrm{O}_G(2) + \mathrm{O}_G(0)) + \mathrm{O}_G(0), \\ &= c(\not{D}^2_{(2)}) + x\mathrm{O}_G(2) + \mathrm{O}_G(0), \qquad \not{D}^2_{(2)} = H_R + F^{\mathcal{E}}(0),\end{aligned} \tag{3.25}$$

where $H_R = -\sum_{i=1}^n(\partial_i - \frac{1}{4}R_{ij}^M(0)x^j)^2$ and $F^{\mathcal{E}}(0) = \frac{1}{2}F^{\mathcal{E}}(\partial_k, \partial_l)(0)\,\mathrm{d}x^k \wedge \mathrm{d}x^l$. In particular, we see that $\not{D}^2$ has Getzler order 2.

Lemma 3.4. *Any Volterra ΨDO parametrix Q for $\not{D}^2 + \partial_t$ is of the form*

$$Q = c(Q_{(-2)}) + x\mathrm{O}_G(-2) + \mathrm{O}_G(-4), \qquad Q_{(-2)} = (H_R + F^{\mathcal{E}}(0) + \partial_t)^{-1}. \tag{3.26}$$

In particular Q has Getzler order -2.

Proof. Thanks to ([20], Lemma 5) we know that Q has Getzler order -2 and model operator $Q_{(-2)} = (H_R + F^{\mathcal{E}}(0) + \partial_t)^{-1}$. In order to get (3.26) notice that since Q is an inverse for $\not{D}^2 + \partial_t$ modulo $\Psi_{\mathrm{v}}^{-\infty}(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p) \otimes \mathrm{Cl}(n)$, hence modulo operators of Getzler orders $-\infty$, we have

$$Q(\not{D}^2 + \partial_t)c(Q_{(-2)}) = (1 + \mathrm{O}_G(-\infty))c(Q_{(-2)}) = c(Q_{(-2)}) + \mathrm{O}_G(-\infty). \tag{3.27}$$

On the other hand, from (3.21) and (3.25) we get

$$\begin{aligned}c(Q_{(-2)})(\not{D}^2 + \partial_t) &= c((\not{D}^2_{(2)} + \partial_t)^{-1})(c(\not{D}^2_{(2)} + \partial_t) + x\mathrm{O}_G(2) + \mathrm{O}_G(0)) \\ &= 1 + x\mathrm{O}_G(0) + \mathrm{O}_G(-2).\end{aligned} \tag{3.28}$$

Therefore, we also obtain

$$Q(c(\not{D}^2_{(2)} + \partial_t) + \mathrm{O}_G(0))c((\not{D}^2_{(2)} + \partial_t)^{-1}) = Q + x\mathrm{O}_G(-2) + \mathrm{O}_G(-4). \tag{3.29}$$

Comparing this to (3.27) gives $Q = c(Q_{(-2)}) + x\mathrm{O}_G(-2) + \mathrm{O}_G(-4)$ as desired. □

Let Q be a Volterra ΨDO parametrix for $\not{D}^2 + \partial_t$. Combining (3.15), (3.21) and (3.26) we get

$$\begin{aligned} \not{D}Q &= (c(\not{D}_{(2)}) + x\mathrm{O}_G(2))(c(Q_{(-2)}) + x\mathrm{O}_G(-2) + \mathrm{O}_G(-4)), \\ &= c(\not{D}_{(2)}Q_{(-2)}) + x^j Q_j + R, \end{aligned} \tag{3.30}$$

with Q_j of Getzler order ≤ 0 and R of Getzler order ≤ -2. Since $K_{x^j Q_j}(0,0,t) = (x^j K_{Q_j})(0,0,t) = 0$, we obtain

$$\sigma[K_{\not{D}Q}(0,0,t)]^{(n)} = K_{\not{D}_{(2)}Q_{(-2)}}(0,0,t)^{(n)} + \sigma[K_R(0,0,t)]^{(n)}. \tag{3.31}$$

On the other hand, as $\not{D}_{(2)}Q_{(-2)}$ is Getzler homogeneous of even degree and R has Getzler order ≤ -2, Lemma 3.2 shows that $K_{\not{D}_{(2)}Q_{(-2)}}(0,0,t)^{(n)} = 0$ and $\sigma[K_R(0,0,t)]^{(n)} = \mathrm{O}(\sqrt{t})$ as $t \to 0^+$. It follows that as $t \to 0^+$ we have

$$\sigma[K_Q(0,0,t)]^{(n)} = K_{\not{D}_{(2)}Q_{(-2)}}(0,0,t) + \sigma[K_R(0,0,t)]^{(n)} = \mathrm{O}(\sqrt{t}). \tag{3.32}$$

Combining this with (3.9) then gives

$$\mathrm{tr}_{\mathcal{E}}\, h_t(x_0, x_0) = \mathrm{O}(\sqrt{t}) \qquad \text{as } t \to 0^+. \tag{3.33}$$

This shows that in the asymptotics (2.12) for $\mathrm{tr}_{\mathcal{E}} h_t(x,x)$ all the coefficients of t^j with $j < \frac{1}{2}$ are in fact zero, that is, we recover Bismut–Freed's asymptotics (3.5).

Acknowledgment

The research of this paper was partially supported by the NSF grant DMS 0409005.

References

[1] M. Atiyah, R. Bott, V. Patodi, On the heat equation and the index theorem, Invent. Math. 19 (1973) 279–330.
[2] M. Atiyah, I. Singer, The index of elliptic operators. I, Ann. Math. 87 (2) (1968) 484–530.
[3] M. Atiyah, I. Singer, The index of elliptic operators. III, Ann. Math. 87 (2) (1968) 546–604.
[4] M. Atiyah, V. Patodi, I. Singer, Spectral asymmetry and Riemannian geometry. I, Math. Proc. Camb. Philos. Soc. 77 (1975) 43–69.
[5] M. Atiyah, V. Patodi, I. Singer, Spectral asymmetry and Riemannian geometry. III, Math. Proc. Camb. Philos. Soc. 79 (1976) 71–99.
[6] R. Beals, P. Greiner, N. Stanton, The heat equation on a CR manifold, J. Differ. Geom. 20 (1984) 343–387.
[7] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, Springer-Verlag, Berlin, 1992.
[9] J.-M. Bismut, J. Cheeger, η-invariants and their adiabatic limits, J. Am. Math. Soc. 2 (1989) 33–70.
[10] J.-M. Bismut, D. Freed, The analysis of elliptic families, Comm. Math. Phys. 107 (1986) 103–163.
[11] J. Brüning, M. Lesch, On the eta-invariant of certain nonlocal boundary value problems, Duke Math. J. 96 (2) (1999) 425–468.
[12] A. Connes, H. Moscovici, The local index formula in noncommutative geometry, Geom. Funct. Anal. 5 (2) (1995) 174–243.
[13] E. Getzler, A short proof of the local Atiyah–Singer index theorem, Topology 25 (1986) 111–117.
[14] P.B. Gilkey, The residue of the local eta function at the origin, Math. Ann. 240 (2) (1979) 183–189.
[15] P.B. Gilkey, The residue of the global η function at the origin, Adv. Math. 40 (3) (1981) 290–307.
[16] P.B. Gilkey, Invariance theory, the heat equation, and the Atiyah–Singer index theorem, Studies in Advanced Mathematics, second ed., CRC Press, Boca Raton, FL, 1995.
[17] P. Greiner, An asymptotic expansion for the heat equation, Arch. Rational Mech. Anal. 41 (1971) 163–218.
[18] R. Melrose, The Atiyah–Patodi–Singer index theorem, A.K. Peters, Boston, 1993.
[19] A. Piriou, Une classe d'opérateurs pseudo-différentiels du type de Volterra, Ann. Inst. Fourier 20 (1970) 77–94.

[20] R. Ponge, A new short proof of the local index formula and some of its applications, Comm. Math. Phys. 241 (2003) 215–234; Erratum, Comm. Math. Phys. 248 (2004) 639.
[21] R. Ponge, On the asymptotic completeness of the Volterra calculus, J. Anal. Math. 94 (2004) 249–263 (Appendix by H. Mikaleyan and R. Ponge).
[22] R. Ponge, Spectral asymmetry, zeta functions and the noncommutative residue. E-print, arXiv, Oct. 03.
[23] M. Rumin, Formes différentielles sur les variétés de contact, J. Differ. Geom. 39 (2) (1994) 281–330.
[24] M. Rumin, Sub-Riemannian limit of the differential form spectrum of contact manifolds, Geom. Funct. Anal. 10 (2000) 407–452.
[25] M. Wodzicki, Spectral asymmetry and zeta functions, Invent. Math. 66 (1982) 115–135 (Corrigenda in [26, 1.25]).
[26] M. Wodzicki, Local invariants of spectral asymmetry, Invent. Math. 75 (1984) 143–177.